Exercise sheet 7

For this exercise sheet, k is a field and $\overline{k}$ is its algebraic closure. We look at some basic descriptions and properties of $\mathrm{SL}_2(k)$ (more precisely $\mathrm{SL}_2(\overline{k})$).

1. Show that any $g \in \mathrm{SL}_2(\overline{k})$ is either:

- Central $\iff g = \pm\mathrm{Id}_2$
- Semisimple $\iff \mathrm{tr(g)} \neq \pm 2 \iff g$ has two distinct eigenvalues $\iff g$ conjugate to a diagonal matrix with distinct entries;
- Regular unipotent $\iff \mathrm{tr}(g) = \pm 2$ and $g \neq \pm\mathrm{Id}_2 \iff g$ is conjugate to a matrix of the form $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, x \in \overline{k}^{\times}$.

2. Given $g \in \mathrm{SL}_2(\overline{k})$, denote by $\mathrm{Cent}_g(\overline{k})$ its centralizer. Show that:

(i) Show that if g is semisimple, then $\mathrm{Cent}_g(\overline{k}) := T_g$ (*maximal torus*) is conjugate to $T = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} : t \in \overline{k} \right\}$.

(ii) If g is regular unipotent, then $\mathrm{Cent}_g(\overline{k}) := \pm N_g$ (*unipotent subgroup*), where N_g is conjugate to $N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \overline{k} \right\}$.

(iii) If we denote by B_g (*Borel subgroup*) the normalizer of N_g, then B_g is conjugate to $B = \left\{ \begin{pmatrix} t & x \\ 0 & t^{-1} \end{pmatrix} : t \in \overline{k}^{\times}, x \in \overline{k} \right\}$.

3. Given $g \in \mathrm{SL}_2(\overline{k})$, let $\mathrm{Conj}(g) = \{hgh^{-1} : h \in \mathrm{SL}_2(\overline{k})\}$. Show that:

- If $g = \pm\mathrm{Id}_2$, then $\mathrm{Conj}(g) = \{g\}$.
- If g regular unipotent, then $\mathrm{Conj}(g)$ is the set of all regular unipotent elements.
- If g semisimple, then $\mathrm{Conj}(g) = \{h \in \mathrm{SL}_2(\overline{k}) : \mathrm{tr}(h) = \mathrm{tr}(g)\}$.

4. (i) Show that $\mathrm{SL}_2(\overline{k})$ acts on $\mathbb{P}^1(\overline{k})$ by fractional linear transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z := \begin{cases} \frac{az+b}{cz+d}, & z \neq -d/c; \\ \infty, & z = -d/c. \end{cases}$$

(ii) Show that the stabilizer of a point is a Borel subgroup, and conversely a Borel subgroup is the stabilizer of a unique point. In particular, deduce that if B is a Borel subgroup, then $\mathbb{P}^1(\overline{k}) \simeq \mathrm{SL}_2(\overline{k})/B$.

(iii) The action is 2-transitive: for all $z_1 \neq z_2 \in \mathbb{P}^1(\overline{k})$, there exists a unique $g \in \mathrm{SL}_2(\overline{k})$ such that $gz_1 = 0, gz_2 = \infty$.

(iv) The pointwise stabilizer of two distinct points $z_1, z_2 \in \mathbb{P}_1(\overline{k})$ is a maximal torus (the intersection of two Borel subgroups is a maximal torus).

(v) The pointwise stabilizer of three distinct points in $\mathbb{P}^1(\overline{k})$ is $\{\pm \mathrm{Id}_2\}$.

5. Show that there exists an absolute constant $D \geq 1$ such that if k is a finite field with $|k| \geq D$, for any Borel subgroup $B \subset \mathrm{SL}_2(\overline{k})$, there exists $g \in \mathrm{SL}_2(k)$ such that $gBg^{-1} \neq B$.

6. Prove statements 6.24-6.29 from the lecture notes.